Classification of multidimensional spacetimes

PAWEŁ TURKOWSKI
Cracow Group of Cosmology
Academy of Agriculture, Physics Dept.,
Mickiewicza 21, 31 - 120 Krakow, Poland

Abstract. *We classify all eleven-dimensional spatially homogeneous spacetimes, which contain a seven-dimensional compact subspace and admit a simple-transitive group of isometry, by enumerating the real ten-dimensional Lie algebras which contain an n-dimensional ($n \geqslant 7$) compact Lie subalgebra. The main result of this paper consists in giving a complete list of the distinct real ten-dimensional algebras which admit a non-trivial Levi decomposition. It is hoped that this investigation may be of some help in studying eleven-dimensional and ten-dimensional cosmologies which are considered in the context of Kaluza-Klein, supergravity and superstring theories.*

1. INTRODUCTICN

The idea that the observable world is a part of a higher-dimensional manifold, of which the extra spatial dimensions form a compact manifold whose size is estremly small, is a promising approach to the unification of all the fundamental forces of nature. If the extra dimensions of the world are, to be treated as a reality, a drastically different character of the macro and micro-spaces is a cosmological conundrum [3, 17, 18]. It migth be that we just have to accept it as an inexplicable fact; on the other hand, one can look for a driving force which compelled the originally multidimensional universe to hide some of its dimensions.

There is a generally accepted opinion that the answer lies in the Einstein

Key-Words: Lie algebras, classification.
1980 Mathematics Subject Classification: 17 B ϕ 5/83 E 15.

equations. In fact, there is no incompatibility between the fact that the universe is apparently four-dimensional, and the idea that it originated from a multi-dimensional phase. The desired four-dimensional final stage is hoped to be achieved via Einstein equations in some simple, e.g. (Friedman-Robertson-Walker) × (n-sphere) models, thourgh the answer can lie not only in the laws of motion, but also in the initial conditions of the model [20, 21].

Satisfactory though this answer may be, it is not clear what is the role of the cross-product assumption. Clearly, then, to account for the mechanism of the *cosmological dimensional reduction* we have to look at the wider class of the models; the simplest non-cross product spaces, the evolution of which have been analysed, are the $SO(N)$ group manifolds [7].

It would be very desirable to give the full list of the multidimensional models. Similarly as in the case of the three-dimensional homogeneous spaces, which are classified into the nine Bianchi types, the classification of the multidimensional homogeneous Riemannian spaces is based on the list of the algebras of the Killing vectors which generate the isometry group of the given space. The dimension and the structure of these algebras are related to the dimension and topology of the space on which the symmetry group acts. In the previous paper [5] we have classified ten-dimensional real Lie algebras which contain a seven-dimensional compact subalgebras, under the assumption that the algebras are decomposable into the direct sum $L_3 \oplus L_7$ (from now on, the algebra L_r means the algebra of dimension r). These algebras can be used as the algebras of the isometry groups $G_3 \times G_7$ which act simply-transitively on the spacelike ten-dimensional sections of the eleven-dimensional spacetimes. The dimension 11 is singled out by the realistic Kaluza-Klein theories. Due to the compactness of the extra seven dimensions, the isometry group G_7 and the algebra L_7 must be compact, i.e. Abelian, or one of the following direct sums of Abelian and simple $so(3)$ algebras: $L_1 \oplus so(3) \oplus so(3)$ or $4L_1 \oplus so(3)$.

In the present paper we shall give a complete description of the ten-dimensional real Lie algebras which contain an n-dimensional ($n \geqslant 7$) compact subalgebra. The enumeration of the distinct ten-dimensional Lie algebras may appear to be too special a problem. However for physical applications it is of interest to classify all real Lie algebras of higher dimensions (see e.g. ref. [16] and [7]). The algebras mentioned above classify all eleven-dimensional spatially homogeneous spacetimes which contain a seven-dimensional compact subspace and admit a simple-transitive group of isometry. The results are summarized in Tables I - II. Only one semisimple compact algebra exists. The solvable algebras containing an Abelian subalgebra are considered in Section 2.2. The decomposable non-solvable algebras are listed in Table I, Sec. 2.3. In Sec. 3, we present the explicit details of the computation of the algebras which admit a non-trivial Levi decomposition. The main result of the paper, the 30 algebras of such a type is given in Table II. In the Appendix we give the similar list of the nine-dimensional Lie algebras which contain an n-dimensional ($n \geqslant 6$) compact subalgebras, this dimension being singled out by the superstring theories.

2. CLASSIFICATION

Levi-Malcev theorem states that an arbitrary Lie algebra consists, in a sense, of a maximal solvable ideal and a semisimple subalgebra [1, 9]. Consequently, Lie algebras fall into the following three categories,

(a) the semisimple type algebras S,

(b) the solvable type algebras N,

(c) and the algebras which admit non-trivial Levi decomposition $L = N +) S$, where +) denotes a semidirect sum. The subalgebra S and the ideal N are called a Levi factor of L and a radical of L, respectively.

Applying the theorem to the 10-dimensional real Lie algebras, we obtain the following conclusions:

2.1. Semisimple algebras

By the Cartan theorem, a Lie algebra is semisimple if and only if it is the direct sum of simple subalgebras. Cartan's classification of all real simple Lie algebras is well known. In the 10-dimensional case there are three semisimple algebras: the algebra of the rotation group in five-dimensional space $so(5)$ and the de Sitter Lie algebras $so(4, 1)$, $so(3, 2)$. Unfortunately, the maximal compact subalgebras of the de Sitter algebras (by virtue of the so called Cartan decomposition: $so(4)$ and $so(3) \oplus so(2)$, respectively [1]) are of dimension less than seven. Hence, $so(5)$ is the only semisimple algebra S_{10} which is appropriate as an algebra of the Killing vectors of the 11-dimensional cosmology.

2.2. Solvable algebras

At present, a complete list of solvable Lie algebras is unknown. The number of possible structures rapidly increases as one passes to higher dimensions, and only solvable algebras of dimension up to five have systematically been studied (see Ref. 13, 14, 16, 10). Real nilpotent («nilpotent» implies «solvable») algebras of dimension six are listed by Morozov [12]. Another subclass of the six-dimensional solvable algebras, i.e. those which have a five-dimensional nilpotent ideal, is treated by Mubarakzyanov [15]. Recently, an analysis of the seven-dimensional nilpotent algebras has appeared in the literature [11].

We require N_{10} to possess a seven-dimensional compact subalgebra. A solvable

Lie algebra has no semisimple subalgebras, thus the only admissible compact subalgebras of N is the Abelian algebra. Hence, the algebra N_{10} is an extension of $7L_1$. There is a great number of such decomposable extensions, e.g. $L_3(\mathrm{I}-\mathrm{VII}) \oplus 7L_1$ algebras, where $L_3(\,.\,)$ denotes a Bianchi-type algebra. Following Mubarakzyanov's method, indecomposable solvable algebras can be found, in principle, by considering the dimension of the maximal nilpotent ideals. On the strength of Mubarakzyanov's theorem, the dimension is greater than, or equal to, 5 for the algebra of dimension 10 [13]. However, as far as we know, nilpotent algebras of dimensions 8 - 10 are unknown and the construction of the class of the solvable algebras mentioned above is not possible.

2.3. Semidirect sums of a solvable algebra and a semisimple algebra

There exist six simple Lie algebras of dimension less than ten, namely: 3-dimensional $sl(2,R)$, $so(3)$, 6-dimensional $so(3,1)$, and 8-dimensional $su(3)$, $sl(3,R)$, $su(2,1)$. Consequently, for 10-dimensional algebra, dimension of its radical N has to be equal 1, 2, 4 or 7. Since the maximal compact subalgebras of the $sl(3,R)$ or $su(2,1)$ algebras (which are: $so(3)$, and $so(3) \oplus so(2)$, respectively [1]), have dimensions 3 and 3 + 1, the algebras L_{10}, with $sl(3,R)$ or $su(2,1)$ Levi factor, do not contain a 7-dimensional compact subalgebra. Therefore, we consider the following semidirect sums

(c1) $L_1 +) S_9$, $\quad S_9 = 3so(3)\ (\equiv so(3) \oplus so(3) \oplus so(3))$,

$2so(3) \oplus sl(2,R)$,

$so(3) \oplus 2sl(2,R)$,

$so(3) \oplus so(3,1)$,

$sl(2,R) \oplus so(3,1)$,

(c2) $N_2 +) su(3)$,

(c3) $N_4 +) S_6$, $\quad S_6 = 2so(3), 2sl(2,R)$,

$so(3) \oplus sl(2,R), so(3,1)$,

(c4) $N_7 +) S_3$, $\quad S_3 = so(3), sl(2,R)$.

We endow the simidirect sums (c1) - (c4) with a Lie algebra structure by using $[\ ,\ ]_N$ and $[\ ,\ ]_S$ ie each of these subalgebras. For the Lie brackets between the two subalgerbas, we set

$$(1) \qquad [e_i, e_j] = R(e_i) * e_j, \qquad e_i \in S,\ e_j \in N,$$

where linear mapping $R(e_i) : N \ni e_j \to R(e_i) * e_j \in N$ is a derivation of N:

(2) $$R(e_i) * [e_J, e_K] = [R(e_i) * e_J, e_K] + [e_J, R(e_i) * e_K].$$

The set $\{R(e_i)\}$ forms a Lie algebra itself (the derivation algebra). Furthermore, the Jacobi identity implies that the homomorphism of S into the derivation algebra, $S \ni e_i \to R(e_i)$, must be a representation of the semisimple algebra S by real matrices. These statements reduce the classification problem for $L_{10} = N +) S$ to the one to find all those real irreducible representations of the algebras S_9, S_8, S_6 and S_3 which are the derivations of L_1, N_2, N_4 and N_7 respectively.

It is clear that the zero-matrix representation of S, acting in N is a derivation of N. The zero-matrix representation of S reduces the semidirect sum $N +) S$ to the ordinary direct sum $N \oplus S$. For the convenience of the reader, Table I lists the 10-dimensional algebras that can be written as a direct sum of semisimple and a solvable algebras. The algebras which do not contain an n-dimensional ($n \geqslant 7$) compact subalgebra have been omitted from the list.

Table I. The real Lie algebras of dimension ten which contain an n-dimensional ($n \geqslant 7$) compact subalgebra and which admit the Levi decomposition $L = N \oplus S$.

Type	*10-dimensional algebra*	*The maximal compact subalgebra*
(c1)	$L_1 \oplus S_9$:	
	$S_9 = 3so(3)$,	$L_1 \oplus 3so(3)$,
	$S_9 = 2so(3) \oplus sl(2,R)$,	$2L_1 \oplus 2so(3)$,
	$S_9 = so(3) \oplus so(3,1)$,	$L_1 \oplus 2so(3)$,
(c2)	$N_2 \oplus su(3)$:	
	N_2 – Abelian,	$2L_1 \oplus su(3)$,
	N_2 – solvable: $[e_1, e_2] = e_2$	$L_1 \oplus su(3)$,
(c3)	$N_4 \oplus S_6$:	
	$S_6 = 2so(3)$, N_4 – arbitrary, $mL_1 \subset N_4$, $4 \geqslant m \geqslant 1$,	$mL_1 \oplus 2so(3)$,
	$S_6 = so(3) \oplus sl(2,R)$, $mL_1 \subset N_4$, $4 \geqslant m \geqslant 3$,	$(m+1)L_1 \oplus so(3)$,
	$S_6 = so(3,1)$, $N_4 = 4L_1$	$4L_1 \oplus so(3)$
(c4)	$N_7 \oplus S_3$:	
	$S_3 = so(3)$, $mL_1 \subset N_7$, $7 \geqslant m \geqslant 4$,	$mL_1 \oplus so(3)$,
	$S_3 = sl(2,R)$, $mL_1 \subset N_7$, $7 \geqslant m \geqslant 6$,	$(m+1)L_1$

All we have left to do is enumerate the various possible non-zero representations of the Levi factors mentioned above and calculate the admissible radicals in order to find all possible algebras. This is the subject of the next Section.

3. THE ENUMERATION OF THE DISTINCT REAL TEN-DIMENSIONAL ALGEBRAS WHICH ADMIT A NON-TRIVIAL LEVI DECOMPOSITION

Now, we shall turn to our main task, an examination of the algebras which are not direct sums of semisimple and a solvable algebras of a lower dimension. For later purposes we collect here some facts concerning representations.

THEOREM 1. *Every reducible representation of a semisimple real Lie algebra is completely reducible* [19, 4].

THEOREM 2. *The representations* $\mathbf{D}_J$ $(J = 0, \frac{1}{2}, 1, \ldots)$, *of dimension* $d = 2J + 1$, *form the complete list of irreducible representations of* $sl(2, C)$.

$\mathbf{D}_J$:

$$e_1 \to \begin{pmatrix} 2J & & & \\ & 2J-2 & & \\ & & \ddots & \\ & & & -2J \end{pmatrix},$$

$$e_3 \to \begin{pmatrix} 0 & 2J & & & \\ & 0 & 2J-1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix}, \quad e_3 \to \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 2 & 0 & & \\ & & \ddots & \ddots & \\ & & & 2J & 0 \end{pmatrix}$$

where $\{e_1, e_2, e_3\}$, *with the multiplication table* $[e_1, e_2] = 2e_2$, $[e_1, e_3] = -2e_3$, $[e_2, e_3] = e_1$, *form a basis of* $sl(2, C)$ [19, 6].

Remark. Since the algebra $so(3)$ and $sl(2, R)$ are the real forms of the complex Lie algebra $sl(2, C)$, theorem 2 provides (real) representations of $sl(2, R)$ and (complex) representations of $so(3)$; these are the matrices: $\mathbf{D}^J(\bar{e}_1) \equiv \frac{i}{2}\, \mathbf{D}_J(e_2) + \frac{i}{2}\, \mathbf{D}_J(e_3)$, $\mathbf{D}^J(\bar{e}_2) \equiv -\frac{1}{2}\, \mathbf{D}_J(e_2) + \frac{1}{2}\, \mathbf{D}_J(e_3)$, $\mathbf{D}^J(\bar{e}_3) \equiv \frac{i}{2}\, \mathbf{D}_J(e_1)$, where $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ forms a basis of $so(3)$.

Fortunately, there is a fundamental theorem, due to Cartan, connecting complex irreducible representations and real irreducible representations (see

Ref. [2], [8]). For the three, five and seven-dimensional representations $\mathbf{D}^J$ ($J = 1, 2, 3$) a trasformation matrix $\mathbf{U}$ can be found such that the representation $\mathbf{R}_{2J+1} = \mathbf{U}\mathbf{D}^J\mathbf{U}^{-1}$ is real [4]. For example, the adjoint representation (**ad**) of $so(3)$ algebra is equivalent to the representation $\mathbf{D}^1$. The representations of this type are called, after Iwahori [8], the representations of the *first class* and denoted $\mathbf{R}_3^I$ (which is equivalent to **ad** $so(3)$), $\mathbf{R}_5^I$ and $\mathbf{R}_7^I$, respectively. To «realify» the complex representation $\mathbf{D}^{1/2}$ amounts to treating the complex matrices $A + i\mathbf{B}$ as the real matrices $\begin{pmatrix} \mathbf{A} & -\mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{pmatrix}$ of the double size. Then we have an irreducible real representation of the *second class* $\mathbf{R}_4^{II}$.

The Cartan's theorem implies that the real irreducible representations of $so(3)$: **ad** $so(3)$, $\mathbf{R}_4^{II}$, $\mathbf{R}_5^I$, $\mathbf{R}_7^I$, are the only one of dimension 7 or less. We will not present the method of finding the matrices $\mathbf{U}$ mentioned above but simply give the explicit matrix representations: $\mathbf{R}_4^{II}$, $\mathbf{R}_5^I$, $\mathbf{R}_7^I$. These are the following.

$\mathbf{R}_4^{II}$:

$$\bar{e}_1 \to \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}, \quad \bar{e}_2 \to \begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix},$$

$$\bar{e}_3 \to \begin{pmatrix} 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \end{pmatrix}$$

$$\mathbf{R}_5^I:\quad e_1 \to \begin{pmatrix} 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & -\frac{3}{2} \\ \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{pmatrix}, \quad e_2 \to \begin{pmatrix} 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & -\frac{3}{2} \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix},$$

$$e_3 \to \begin{pmatrix} 0 & -2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{R}_7^I:\quad e_1 \to \begin{pmatrix} 0 & 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & -\frac{5}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{5}{2} & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 \end{pmatrix},$$

$$e_2 \to \begin{pmatrix} 0 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & -\frac{5}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & -\frac{5}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 \end{pmatrix}$$

$$e_3 \to \begin{pmatrix} 0 & -3 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

THEOREM 3. *Let L be the direct sum of two semisimple algebras L' and L''. Then any irreducible representation R of L is equivalent to the tensor product of irreducible representations R' and R'' of L' and L''* [19].

A consequence of theorems 1 - 3 is that there is no non-zero 1×1 representation of semisimple S_9 algebras. Also the algebra $su(3)$ has no non-zero 2×2 real representation. To show how the method does work, we give the calculations in details for $L_{10} = N_4 \dot{+}) 2so(3)$. This is an example of case (c3), Sec. 2.3.

The algebra $2so(3)$ is defined by the following non-zero commutation relations of the basis elements

$$(3) \qquad \begin{aligned} [e_i, e_j] &= \epsilon_{ijk} e_k \qquad (\text{for } i, j, k = 1, 2, 3), \\ [e_p, e_q] &= \epsilon_{pqr} e_r \qquad (\text{for } p, q, r = 4, 5, 6), \end{aligned}$$

where $\{e_1, \ldots, e_6\}$ are the basis elements. Since $so(3)$ has a representation by 3×3 matrices (the adjoined representation) and none of smaller degree than 3, we can set

$$R(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, R(e_2) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, R(e_3) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$R(e_4) = R(e_5) = R(e_6) = 4 \times 4$ zero-matrix.

This, via relation (1), implies the following commutation relations

$$(4) \qquad \begin{aligned} [e_1, e_8] &= +e_9, \quad [e_2, e_7] = -e_9, \quad [e_3, e_7] = +e_8 \\ [e_1, e_9] &= -e_8, \quad [e_2, e_9] = +e_7, \quad [e_3, e_8] = -e_7 \end{aligned}$$

where $\{e_7, e_8, e_9, e_{10}\}$ are the basis elements of N_4. Jacobi identity (2) implies that the 4-dimensional solvable algebra N_4 is Abelian (all the rest commutators are equal to zero) or it is a solvable algebra given by the following non-zero commutators

(5) $[e_J, e_{10}] = e_J, \qquad J = 7, 8, 9.$

This is the algebra $A^{1,1}_{4,5}$ in the list by Patera at al. [16], or the algebra $g_{4,5}(\beta = \gamma = 1)$ in Mubarakzyanov's notation [13]. Hence, there are the following two algebras defined by eqs. (3 - 4) and (3 - 5), respectively: $L_{10,1} = (4L_1 +) so(3)) \oplus so(3)$ and $L_{10,2} = (A^{1,1}_{4,5} +) so(3)) \oplus so(3)$. The notation is that $L_{r,j}$ is the j^{th} type of an r-dimensional algebra.

The seven-dimensional compact subalgebras of the algebra $L_{10,1}$ are $L_1 \oplus 2so(3)$ and $4L_1 \oplus so(3)$. The elements of the compact subalgebra $L_1 \oplus 2so(3)$ are linear combinations of $\{e_1, \ldots, e_6, e_{10}\}$, whereas the subalgebra $4L_1 \oplus so(3)$ is spanned by $\{e_4, \ldots, e_{10}\}$. The seven-dimensional compact subalgebra of the algebra $L_{10,2}$ is $L_1 \oplus 2so(3)$. The elements of the compact subalgebra are linear combinations of $\{e_1, \ldots, e_6, e_{10}\}$.

In all cases (c3) - (c4), Sec. 2.3, we proced analogously; the results are presented in Table II. We present the radicals N, the Levi factors S and the representation of

Table II. The list of ten-dimensional real Lie algebras which contain an n-dimensional ($n \geqslant 7$) compact subalgebra and which admit the Levi decomposition $L = N +) S$.

Name	*The Levi decomposition*	*The representation of the Levi factor*
$L_{10,1}$	$L_1 \oplus (3L_1 +) so(3)) \oplus so(3)$	ad $so(3)$
$L_{10,2}$	$(A^{1,1}_{4,5} +) so(3)) \oplus so(3)$	ad $so(3) \oplus [0]$
$L_{10,3}$	$(4L_1 +) so(3)) \oplus so(3)$	$\mathbf{R}^{II}_4$
$L_{10,4}$	$2L_1 \oplus (2L_1 +) sl(2,R)) \oplus so(3)$	$\mathbf{D}_{1/2}$
$L_{10,5}$	$L_1 \oplus (A_{3,1} +) sl(2,R)) \oplus so(3)$	$\mathbf{D}_{1/2} \oplus \mathbf{D}_0$
$L_{10,6}$	$L_1 \oplus (3L_1 +) sl(2,R)) \oplus so(3)$	$\mathbf{D}_1$
$L_{10,7}$	$(4L_1 +) sl(2,R)) \oplus so(3)$	$\mathbf{D}_{3/2}$
$L_{10,8}$	$(4L_1 +) sl(2,R)) \oplus so(3)$	$\mathbf{D}_{1/2} \oplus \mathbf{D}_{1/2}$
$L_{10,9}$	$4L_1 \oplus (3L_1 +) so(3))$	ad $so(3)$
$L_{10,10}$	$3L_1 \oplus (A^{1,1}_{4,5} +) so(3))$	ad $so(3) \oplus [0]$
$L_{10,11}$	$3L_1 \oplus (4L_1 +) so(3))$	$\mathbf{R}^{II}_4$
$L_{10,12}$	$2L_1 \oplus (5L_1 +) so(3))$	$\mathbf{R}^{I}_5$
$L_{10,13}$	$L_1 \oplus (6L_1 +) so(3))$	ad $so(3) \oplus$ ad $so(3)$

Table II. (continued)

Name	*The Levi decomposition*	*The representation of the Levi factor*
$L_{10,14}$	$7L_1 +) so(3)$	$\mathbf{R}_7^I$
$L_{10,15}$	$7L_1 +) so(3)$	$\mathbf{R}_4^{II} \oplus \mathbf{ad}\, so(3)$
$L_{10,16}$	$5L_1 \oplus (2L_1 +) sl(2,R))$	$\mathbf{D}_{1/2}$
$L_{10,17}$	$4L_1 \oplus (A_{3,1} +) sl(2,R))$	$\mathbf{D}_{1/2} \oplus \mathbf{D}_0$
$L_{10,18}$	$4L_1 \oplus (3L_1 +) sl(2,R))$	$\mathbf{D}_1$
$L_{10,19}$	$3L_1 \oplus (4L_1 +) sl(2,R))$	$\mathbf{D}_{3/2}$
$L_{10,20}$	$3L_1 \oplus (4L_1 +) sl(2,R))$	$\mathbf{D}_{1/2} \oplus \mathbf{D}_{1/2}$
$L_{10,21}$	$2L_1 \oplus (5L_1 +) sl(2,R))$	$\mathbf{D}_2$
$L_{10,22}$	$2L_1 \oplus (5L_1 +) sl(2,R))$	$\mathbf{D}_1 \oplus \mathbf{D}_{1/2}$
$L_{10,23}$	$L_1 \oplus (6L_1 +) sl(2,R))$	$\mathbf{D}_{5/2}$
$L_{10,24}$	$L_1 \oplus (6L_1 +) sl(2,R))$	$\mathbf{D}_{3/2} \oplus \mathbf{D}_{1/2}$
$L_{10,25}$	$L_1 \oplus (6L_1 +) sl(2,R))$	$\mathbf{D}_{1/2} \oplus \mathbf{D}_{1/2} \oplus \mathbf{D}_{1/2}$
$L_{10,26}$	$L_1 \oplus (6L_1 +) sl(2,R))$	$\mathbf{D}_1 \oplus \mathbf{D}_1$
$L_{10,27}$	$7L_1 +) sl(2,R)$	$\mathbf{D}_3$
$L_{10,28}$	$7L_1 +) sl(2,R)$	$\mathbf{D}_2 \oplus \mathbf{D}_{1/2}$
$L_{10,29}$	$7L_1 +) sl(2,R)$	$\mathbf{D}_{3/2} \oplus \mathbf{D}_1$
$L_{10,30}$	$7L_1 +) sl(2,R)$	$\mathbf{D}_1 \oplus \mathbf{D}_{1/2} \oplus \mathbf{D}_{1/2}$

the algebra S determining the semidirect sum (as defined in Equation (1)). The following notation is used: $so(3)$ and $sl(2, R)$ denote the simple three-dimensional algebras. The Abelian n-dimensional algebra is denoted by nL_1. The basic commutation relations of the algebras $A_{3,1}$ and $A_{4,5}^{1,1}$ are $[e_2, e_3] = e_1$ and $[e_i, e_4] = e_i$ (for $i = 1, 2, 3$), respcectively.

A number of algebras which do not contain an n-dimensional ($n \geqslant 7$) compact subalgebra do exist. This is demonstrated by the following example. Consider the semidirect sum of an N_3 and the $sl(2, R)$ algebras defined by the representation $\mathbf{D}_{1/2} \oplus \mathbf{D}_0$ of $sl(2, R)$:

$$\begin{aligned} &[e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_2, e_3] = e_1, \\ &[e_1, e_4] = e_4, \quad [e_2, e_5] = e_4, \quad [e_3, e_4] = e_5, \qquad (6) \\ &[e_1, e_5] = -e_5, \end{aligned}$$

where $\{e_1, e_2, e_3\}$ are the basis elements of $sl(2,R)$ and N_3 is spanned by $\{e_4, e_5, e_6\}$. It can be easily verified that condition (2) is satisfied if and only if $N_3 = 3L_1$, $N_3 = A_{3,1}$, where $A_{3,1}$ is defined by $[e_4, e_5] = e_6$ or $N_3 = A_{3,3}$, where $A_{3,3}$ is defined by $[e_4, e_6] = e_4$, $[e_5, e_6] = e_5$. This provides the ten-dimensional algebras $L_{10,4}$, $L_{10,5}$ and $L_{10} = L_1 \oplus (A_{3,3} +) sl(2,R)) \oplus so(3)$, respectively. However, the last-mentioned algebra does not contain a 7-dimensional compact subalgebra. The algebras of this type have been omitted from Table II.

ACKNOWLEDGEMENTS

I am thankful to M. Demianski for suggesting the problem and to M. Heller for reading the manuscript.

This work was supported in part by the Polish Interdisciplinary Research Project CPBP-01-03.

APPENDIX

We give the list of nine-dimensional real Lie algebras which contain a compact subalgebra of dimension $n \geqslant 6$. There are three semisimple algebras which possess the above property: the compact algebra $3so(3)$ and the algebras $2so(3) \oplus sl(2,R)$, $so(3) \oplus so(3,1)$ which contain $L_1 \oplus 2so(3)$ and $2so(3)$ as the maximal compact subalgebras, respectively. In Table III the algebras which admit the Levi decompo-

Table III. The list of 9-dimensional real Lie algebras which contain an n-dimensional ($n \geqslant 6$) compact subalgebra and which admit the Levi decomposition $L = N \oplus S$.

Type	*9-dimensional algebra*	*The maximal compact subalgebra*
(c1)	$L_1 \oplus su(3)$,	$L_1 \oplus su(3)$
(c2)	$N_3 \oplus S_6$:	
	$S_6 = 2so(3)$, N_3 – arbitrary,	$2L_1 \oplus 2so(3)$
	$S_6 = so(3) \oplus sl(2,R)$, N_3 – arbitrary,	$3L_1 \oplus so(3)$,
	$3L_1 \oplus so(3,1)$,	$3L_1 \oplus so(3)$,
(c3)	$N_6 \oplus S_3$:	
	$S_3 = sl(2,R)$, $mL_1 \subset N_6$, $6 \geqslant m \geqslant 5$,	$(m+1)L_1$,
	$S_3 = so(3)$, $mL_1 \subset N_6$, $6 \geqslant m \geqslant 3$,	$mL_1 \oplus so(3)$.

sition $L = N \oplus S$ are given. There are 20 algebras which admit a non-trivial Levi decomposition $L = N +) S$. Their structure is presented in Table IV. These algebras are contained, as subalgebras, in the ten-dimensional algebras listed in Table II; for the commutation rules see Section 3.

Table IV. The list of nine-dimensional real Lie algebras which contain an n-dimensional ($n \geq 6$) compact subalgebra and which admit the Levi decomposition $L = N +) S$.

Name	*The Levi decomposition*	*The representation of the Levi factor*
$L_{9,1}$	$(3L_1 +) so(3)) \oplus so(3)$	ad $so(3)$
$L_{9,2}$	$L_1 \oplus (2L_1 +) sl(2,R)) \oplus so(3)$	$\mathbf{D}_{1/2}$
$L_{9,3}$	$(A_{3,1} +) sl(2,R)) \oplus so(3)$	$\mathbf{D}_{1/2} \oplus \mathbf{D}_0$
$L_{9,4}$	$(3L_1 +) sl(2,R)) \oplus so(3)$	$\mathbf{D}_1$
$L_{9,5}$	$3L_1 \oplus (3L_1 +) so(3))$	ad $so(3)$
$L_{9,6}$	$2L_1 \oplus (A_{4,5}^{1,1} +) so(3))$	ad $so(3) \oplus [0]$
$L_{9,7}$	$2L_1 \oplus (4L_1 +) so(3))$	$\mathbf{R}_4^{II}$
$L_{9,8}$	$L_1 \oplus (5L_1 +) so(3))$	$\mathbf{R}_5^{I}$
$L_{9,9}$	$6L_1 +) so(3)$	ad $so(3) \oplus$ ad $so(3)$
$L_{9,10}$	$4L_1 \oplus (2L_1 +) sl(2,R))$	$\mathbf{D}_{1/2}$
$L_{9,11}$	$3L_1 \oplus (A_{3,1} +) sl(2,R))$	$\mathbf{D}_{1/2} \oplus \mathbf{D}_0$
$L_{9,12}$	$3L_1 \oplus (3L_1 +) sl(2,R))$	$\mathbf{D}_1$
$L_{9,13}$	$2L_1 \oplus (4L_1 +) sl(2,R))$	$\mathbf{D}_{3/2}$
$L_{9,14}$	$2L_1 \oplus (4L_1 +) sl(2,R))$	$\mathbf{D}_{1/2} \oplus \mathbf{D}_{1/2}$
$L_{9,15}$	$L_1 \oplus (5L_1 +) sl(2,R))$	$\mathbf{D}_2$
$L_{9,16}$	$L_1 \oplus (5L_1 +) sl(2,R))$	$\mathbf{D}_1 \oplus \mathbf{D}_{1/2}$
$L_{9,17}$	$6L_1 +) sl(2,R)$	$\mathbf{D}_{5/2}$
$L_{9,18}$	$6L_1 +) sl(2,R)$	$\mathbf{D}_{3/2} \oplus \mathbf{D}_{1/2}$
$L_{9,19}$	$6L_1 +) sl(2,R)$	$\mathbf{D}_{1/2} \oplus \mathbf{D}_{1/2} \oplus \mathbf{D}_{1/2}$
$L_{9,20}$	$6L_1 +) sl(2,R)$	$\mathbf{D}_1 \oplus \mathbf{D}_1$

REFERENCES

[1] A.O. BARUT, R. RACZKA, *Theory of Group Representations and Applications*, PWN-Polish Scientific Publishers, Warsaw, 1977

[2] E. CARTAN, *Les groupes projectifs continus réels qui ne laissent invariante aucune multipicité plane*, Journ. Math. Pures et appl., **10**, 1914, 149 - 186; reprinted in: E. Cartan, Oeuvres complètes, Part 1, Vol. 1, Gauthier-Villars, Paris, 1952, pp. 493 - 530.

[3] A. CHODOS, S. DETWEILER, *Where has the fifth dimension gone?*, Phys. Rev. **D21**, n. 8, 1980, 2167 - 2170.

[4] J.F. CORNWELL. *Group Theory in Physics*, Academic Press, London. 1984.

[5] M. DEMIANSKI, Z. GOLDA, L.M. SOKOLOWSKI, M. SZYDLOWSKI, P. TURKOWSKI, *The Group Theoretical Classification of the 11-dimensional Classical Homogeneous Kaluza-Klein Cosmologies*, J. Math. Phys. **28**, n. 1, 1987. 171 - 173.

[6] J.E. HUMPHREYS, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, New York, 1972.

[7] H. ISHIHARA, *Towards Higher-Dimensional Homogeneous Cosmologies*, Prog. Theor. Phys. **74**, n. 3, 1985, 490 - 500.

[8] N. IWAHORI, *On Real Irreducible Representations of Lie Algebras*, Negoya Math. J. **14**, 1959, 59 - 83.

[9] N. JACOBSON, *Lie Algebras*, Interscience Publishers, New York, 1962.

[10] M.A.H. MACCALLUM, *On the Classification of the Real four-dimensional Lie Algebras*, Preprint Queen Mary College, London, undated.

[11] L. MAGNIN, *Sur les algèbras de Lie nilpotentes de dimension* $\leqslant 7$, Journal of Geometry and Physics **3**, n. 1, 1986, 119 - 144.

[12] V.V. MOROZOV, *Classification of the 6-dimensional Nilpotent Lie Algebras* (in Russian), Izv. Vysshikh Uchebn. Zavedenii Mat. 5, n. 4, 1958, 161 - 171.

[13] G.M. MUBARAKZYANOV, *On Solvable Lie Algebras* (in Russian), Izv. Vysshikh Uchebn. Zavedenii Mat. **32**, n. 1, 1963, 114 - 123.

[14] G.M. MUBARAKZYANOV, *Classification of the 5-dimensional Real Lie Algebras* (in Russian), Izv. Vysshikh Uchebn. Zavedenii Mat. **34**, n. 3, 1963, 99 - 106.

[15] G.M. MUBARAKZYANOV, *Classification of the 6-dimensional Solvable Lie Algebras which possess a Nilpotent Basis Element* (in Russian), Izv. Vysshikh Uchebn. Zavedenii Mat. **35**, n. 4, 1963, 104 - 116.

[16] J. PATERA, R.T. SHARP, P. WINTERNITZ, *Invariants of Real Low Dimensional Lie Algebras*, J. Math. Phys. **17**, n. 6, 1976, 986 - 994.

[17] D. SAHDEV, *Towards a Realistic Kaluza-Klein Cosmology*, Phys. Lett. **B137**, n. 3, 4, 1984, 155 - 159.

[18] D. SAHDEV, *Perfect-fluid Higher-dimensional Cosmologies*, Phys. Rev. **D30**, n. 12, 1984, 2495 - 2507.

[19] H. SAMELSON, *Notes on Lie Algebras*, Van Nostrand Reinhold Company, New York, 1969.

[20] A. TOMIMATSU, H. ISHIHARA, *Dimensional Reduction in an Oscillatory Kaluza-Klein Cosmology*, Gen. Rel. Grav. **18**, n. 2, 1986, 161 - 171.

[21] P. TURKOWSKI, K. MASLANKA, *Phase Space Classification of Kaluza-Klein Cosmologies*, Gen Rel. Grav. **19**, n. 6, 1987, 611 - 621.

Manuscript received: June 2, 1987.